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# Self-Dual Chern-Simons Solitons and Generalized Heisenberg Ferromagnet Models

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## ABSTRACT

We consider the (2+1)-dimensional gauged Heisenberg ferromagnet model coupled with the Chern-Simons gauge fields. Self-dual Chern-Simons solitons, the static zero energy solution saturating Bogomol'nyi bounds, are shown to exist when the generalized spin variable is valued in the Hermitian symmetric spaces  $G/H$ . By gauging the maximal torus subgroup of  $H$ , we obtain self-dual solitons which satisfy vortex-type nonlinear equations thereby extending the two dimensional instantons in a nontrivial way. An explicit example for the  $CP(N)$  case is given.

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Recently, there appeared an action principle of the generalized Heisenberg ferromagnet model in terms of a nonrelativistic nonlinear sigma model defined on a Lie group  $G$  [1]. This action possesses a local  $H$  subgroup symmetry so that the physical spin variables take value on the coadjoint orbit of the Hermitian symmetric space  $G/H$ . The symplectic structure on each orbit also allows a direct first order action in terms of generalized spin variables. The use of Hermitian symmetric space made possible a systematic generalization of the Heisenberg ferromagnet model according to the Cartan's classification of symmetric spaces [2] and led to the infinite conservation laws of the model [1].

In this Letter, we consider the generalized Heisenberg ferromagnet model in (2+1)-dimensions. The motivations are twofold. Firstly, the model itself can be used in describing generalized planar ferromagnetisms where the generality coming from the large degrees of freedom of symmetric spaces can be used to handle various physical situations. Secondly, by gauging and coupling the model with the Chern-Simons gauge fields, we are led to the Chern-Simons self-dual solitons [3] which attracted an upsurge of recent interests in regard of the application to the quantum Hall effect and the high-Temperature superconductivity [4, 5, 6]. Here, we focus on the second motivation and show that, using the properties of Hermitian symmetric spaces, the Hamiltonian of the model is bounded below by a topological charge. The resulting Chern-Simons solitons satisfy a vortex-type equation when the model is gauged with the maximal torus subgroup of  $H$  and added by a gauge invariant term which induces vacuum symmetry breaking. To our knowledge, this vortex-type equation is new and is likely to possess similar properties to those of the vortex equation of Abelian Higgs model [7] or gauged non-linear Schrödinger model [8]. As an example, we present an explicit expression for the vortex-type equation in the case of  $CP(N)$ .

We first recall that the action principle for the (1+1)-dimensional generalized Heisenberg ferromagnet model defined on the Hermitian symmetric space  $G/H$  can be given by [1]

$$S = \int dt dx [ \text{Tr} (2Kg^{-1}\partial_t g + \partial_x(gKg^{-1})\partial_x(gKg^{-1})) ] \quad (1)$$

where  $g$  is a map  $g : R^{1+1} \rightarrow G$  for the Lie group  $G$  and  $K$  is an element in the Cartan subalgebra of the Lie algebra  $\mathfrak{g}(= \mathfrak{h} \oplus \mathfrak{m})$  whose centralizer in  $\mathfrak{g}$  is  $\mathfrak{h}$ , i.e.  $\mathfrak{h} = \{V \in \mathfrak{g} : [V, K] = 0\}$ . Up to a scaling,  $J \equiv \text{ad}K = [K, *]$  defines a linear map  $J : \mathfrak{m} \rightarrow \mathfrak{m}$

which satisfies the complex structure condition  $J^2 = -1$  or,

$$[K, [K, M]] = -M, \quad (2)$$

for  $M \in \mathfrak{m}$ . The action  $S$  in Eq. (1) possesses local symmetry under  $g \rightarrow gh$  for  $h : R^{1+1} \rightarrow H$  so that the physical variable is effectively given by a generalized spin variable  $Q \equiv gKg^{-1}$  valued in the coadjoint orbit of  $G/H$ .

Now, we extend the model to the (2+1)-dimensional case and introduce gauge fields  $A_\mu$  which gauges the left multiplication of group  $G$ ;  $g \rightarrow g'g$ . Consider the (2+1)-dimensional gauged Heisenberg ferromagnet model defined by the following action;

$$S = \int dt d^2x \left\{ [\text{Tr} (2Kg^{-1}D_tg + D_i(gKg^{-1})D_i(gKg^{-1}))] - V(gKg^{-1}) + \mathcal{L}_{CS} \right\}. \quad (3)$$

We use the convention in which the generators  $T^A$ 's satisfy the commutation relation,  $[T^A, T^B] = f^{ABC} T^C$ , and the normalization,  $\text{Tr}(T^A T^B) = (-1/2)\delta_{AB}$ . The covariant derivative is defined on fundamental and adjoint representations by

$$D_\mu g = \partial_\mu g + A_\mu g, \quad A_\mu = A_\mu^A T^A, \quad D_\mu(gKg^{-1}) = D_\mu Q = [\partial_\mu + A_\mu, Q]. \quad (4)$$

The potential  $V(gKg^{-1})$  is given in terms of the generalized spin  $Q^A = -2\text{Tr}(QT^A)$ :

$$V = \frac{1}{2} I^{AB} Q^A Q^B \quad (5)$$

where  $I^{AB}$  is a constant symmetric matrix measuring the anisotropy of the system [10]. We assume that the dynamics of gauge fields is governed by the Chern-Simons action  $\mathcal{L}_{CS}$ :

$$\mathcal{L}_{CS} = -\kappa \epsilon^{\mu\nu\rho} \text{Tr}(\partial_\mu A_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho). \quad (6)$$

Then, the equations of motion in terms of the generalized spin  $Q$  are the gauged planar Landau-Lifshitz equation;

$$D_t Q + D_i [Q, D_i Q] + [\bar{Q}, Q] = 0 \quad (7)$$

and the gauge field equation;

$$\frac{\kappa}{2} \epsilon^{\mu\nu\rho} F_{\mu\nu}^A = J^{\mu A} \quad (8)$$

where  $\bar{Q} = I^{AB}Q^BT^A$ . The current density is given by

$$J^A = (Q^A, -2\text{Tr}(T^A[Q, D^iQ])). \quad (9)$$

In particular, the zeroth component gives the Gauss's law constraint:

$$G^A = \frac{\kappa}{2}\epsilon^{ij}F_{ij}^A - Q^A = 0 \quad (10)$$

We may rewrite the action Eq.(3) in terms of Hamiltonian

$$S = \int dt d^2x \left\{ \text{Tr} (2Kg^{-1}\dot{g}) - \frac{\kappa}{2}\epsilon^{ij}A_i^A\dot{A}_j^A - \mathcal{H} + A_0^AG^A \right\} \quad (11)$$

where the Hamiltonian is given by

$$H = \int d^2x \mathcal{H} = \int d^2x \left[ \frac{1}{2}(D_iQ^A)^2 + V(Q^A) \right]. \quad (12)$$

Owing to the property of Hermitian symmetric spaces Eq. (2), we have a useful identity;

$$[Q, [Q, D_iQ]] = g[K, [K, [g^{-1}D_ig, K]]]g^{-1} = -g[g^{-1}D_ig, K]g^{-1} = -D_iQ, \quad (13)$$

which brings the Hamiltonian  $H$  into the Bogomol'nyi type;

$$\begin{aligned} H &= \int d^2x \left[ \frac{1}{4}(D_iQ^A \pm \epsilon_{ij}[Q, D_jQ]^A)^2 + V(Q^A) \pm \frac{1}{2}\epsilon_{ij}Q^A[D_iQ, D_jQ]^A \right] \\ &= \int d^2x \left[ \frac{1}{4}(D_iQ^A \pm \epsilon_{ij}[Q, D_jQ]^A)^2 + V(Q^A) \right] \pm \frac{1}{2}\epsilon_{ij}F_{ij}^AQ^A \pm 4\pi T. \end{aligned} \quad (14)$$

The topological charge  $T$  is defined by

$$T = \frac{1}{8\pi} \int d^2x [\epsilon_{ij}Q^A[\partial_iQ, \partial_jQ]^A - 2\epsilon_{ij}\partial_i(Q^AA_j^A)]. \quad (15)$$

Thus, the energy is bounded below by the topological charge  $T$  when the potential  $V$  is chosen such that

$$V \pm \frac{1}{2}\epsilon_{ij}F_{ij}^AQ^A = 0. \quad (16)$$

Or, upon imposing the Gauss's law constraint, it is equivalent to choosing the constant matrix

$$I^{AB} = \mp \frac{2}{\kappa} \delta_{AB}. \quad (17)$$

The minimum energy arises when the spin variable satisfies the first order self-duality equation,

$$D_i Q = \mp \epsilon_{ij} [Q, D_j Q]. \quad (18)$$

By taking a commutator with  $Q$ , we note that Eq. (18) is consistent with Eq. (13). For a given sign of the potential, the energy minimizing solutions are either self-dual or anti self-dual, but not both. As in the case of the non-relativistic non-linear Schrödinger model [8], this is in contrast with the Chern-Simons solitons of the relativistic Abelian Higgs model [7] or the gauged nonlinear sigma model [9] where a fixed potential admits both self-dual and anti self-dual solitons as energy minimizing static configurations. The specific choice of the potential Eq. (5) describes isotropic case where the potential trivially reduces to a constant. However, if we choose the gauge group to be a proper subgroup of  $G$ , then the potential becomes nontrivial and describes anisotropic cases. Another observation to be made is that in the absence of gauge fields the self-duality equation (18) is precisely that of two dimensional instantons in the principle chiral model which has been classified according to each symmetric spaces [11]. The role of the gauge field is that, through the Chern-Simons dynamics, it changes instantons into vortices.

In order to see how vortices arise in our model, we take the gauge group to be the maximal torus subgroup of  $H$  and introduce gauge invariant terms to the action which induce vacuum symmetry breaking. Explicitly, we take  $H^a (a = 1, \dots, \text{rank}(H))$  to be generators of the maximal torus group and add to the action Eq. (3) a linear term

$$\Delta S = \int dt d^2 x A_o^a v^a \quad (19)$$

where each  $v^a$  is a constant and the sum is taken over  $a = 1, \dots, \text{rank}(H)$ . Up to a total derivative term,  $\Delta S$  is invariant under the gauge transform. Then, the gauge fields  $A_\mu = A_\mu^a H^a$  and the Chern-Simons action reduces to a sum of Abelian Chern-Simons terms,

$$\mathcal{L}_{CS} = \frac{\kappa}{2} \epsilon^{\mu\nu\rho} \partial_\mu A_\nu^a A_\rho^a. \quad (20)$$

The Gauss's law is replaced by

$$\frac{\kappa}{2} \epsilon_{ij} F_{ij}^a = Q^a - v^a. \quad (21)$$

Also, we have the topological charge replacing Eq. (15)

$$T = \frac{1}{8\pi} \int d^2x [\epsilon_{ij} Q^A [\partial_i Q, \partial_j Q]^A + 2\epsilon_{ij} \partial_i ((v^a - Q^a) A_j^a)]. \quad (22)$$

We assume the potential  $V$  to be of the form

$$V(gKg^{-1}) = \frac{1}{2} \sum_a I^a (Q^a - v^a)^2, \quad (23)$$

then the Bogomol'nyi bound is established with the choice

$$I^1 = \dots = I^{N-1} = \mp \frac{2}{\kappa}. \quad (24)$$

Note that the potential Eq. (23) is nontrivial unlike the previous case and the nonvanishing constants  $v^a$  breaks the symmetry of the vacuum spontaneously.

In the following, in order to be more explicit, we restrict to the  $CP(N-1) = SU(N)/(SU(N-1) \times U(1))$  case and present a detailed analysis. In this case, the element  $K$  in the Cartan subalgebra is given by  $K = (i/N)\text{diag}(N-1, -1, \dots, -1)$ . Now introduce a parameterization of the group element  $g$  of  $SU(N)$  by an  $N$ -tuple,  $g = (Z_1, Z_2, \dots, Z_N)$ ;  $Z_p \in \mathbb{C}^N$  ( $p, q = 1, \dots, N$ ), such that

$$\bar{Z}_p Z_q = \delta_{pq}, \quad \det(Z_1, Z_2, \dots, Z_N) = 1. \quad (25)$$

Then the generalized spin  $Q$  is given by

$$Q = iZ_1 \bar{Z}_1 - iI. \quad (26)$$

All other  $Z_p$ 's with  $p = 2, \dots, N$  disappear in the expression of  $Q$  due to the particular form of  $K$ . In terms of the Fubini-Study coordinate  $\psi_\alpha$  ( $\alpha = 1, 2, \dots, N-1$ ) [12]:

$$z_1 = \frac{1}{\sqrt{1 + |\psi|^2}}, \quad z_{\alpha+1} = \frac{\psi_\alpha}{\sqrt{1 + |\psi|^2}}, \quad ; \quad Z_1^T = (z_1, z_2, \dots, z_N), \quad (27)$$

we have an equivalent expression of  $Q$  in component,

$$Q^A(\psi, \bar{\psi}) = -2i \sum_{p,q=1}^N \bar{z}_p (T^A)_{pq} z_q. \quad (28)$$

We choose the standard expression for  $T^A$ 's:  $T^A = i\lambda^A/2$  where  $\lambda^A$  is the  $SU(N)$  Gell-Mann matrices [13]. The Cartan subalgebra generators  $H^a$  generating the maximal torus group of  $SU(N-1) \times U(1)$  are given by  $N-1$  diagonal matrices

$$H_{pq}^a = i(\sum_{k=1}^a \delta_{ik}\delta_{jk} - a\delta_{i,a+1}\delta_{j,a+1})/\sqrt{2a(a+1)} ; a = 1, \dots, N-1. \quad (29)$$

Using the complex notation;  $z = x + iy$ ,  $\bar{z} = x - iy$ ,  $A_z = \frac{1}{2}(A_1 - iA_2)$ ,  $A_{\bar{z}} = \frac{1}{2}(A_1 + iA_2)$ , and  $D_z = \frac{1}{2}(D_1 - iD_2)$ ,  $D_{\bar{z}} = \frac{1}{2}(D_1 + iD_2)$ , we obtain an alternative expression of the self-duality equation,

$$D_z Q = \mp i[Q, D_z Q]. \quad (30)$$

With the parameterization of  $Q$  as in Eq. (28), the self-duality equation (30) for the plus sign case becomes a set of  $N-1$  equations,

$$\begin{aligned} (\partial_z + iA_z^1)\bar{\psi}_1 &= 0 \\ (\partial_z + \frac{i}{2}(A_z^1 + \frac{1}{\sqrt{3}}A_z^2))\bar{\psi}_2 &= 0 \\ (\partial_z + \frac{i}{2}(A_z^1 + \frac{1}{\sqrt{3}}A_z^2 + \frac{4}{\sqrt{6}}A_z^3))\bar{\psi}_3 &= 0 \\ &\dots \\ (\partial_z + \frac{i}{2}(A_z^1 + \frac{i}{\sqrt{3}}A_z^2 + \dots + \frac{1}{\sqrt{(N-1)(N-2)/2}}A_z^{N-2} + \frac{N}{\sqrt{N(N-1)/2}}A_z^{N-1}))\bar{\psi}_{N-1} &= 0. \end{aligned} \quad (31)$$

Or, in terms of a shorthand notation

$$D_-^\alpha \equiv \partial_z + \frac{i}{2}(A_z^1 + \frac{1}{\sqrt{3}}A_z^2 + \dots + \frac{1}{\sqrt{(\alpha-1)(\alpha-2)/2}}A_z^{\alpha-2} + \frac{\alpha}{\sqrt{\alpha(\alpha-1)/2}}A_z^{\alpha-1}) \quad (32)$$

we have

$$D_-^\alpha \bar{\psi}_\alpha = 0 ; \alpha = 1, \dots, N-1. \quad (33)$$

Similarly, for the minus sign case, we have

$$D_+^\alpha \equiv \partial_z - \frac{i}{2}(A_z^1 + \frac{1}{\sqrt{3}}A_z^2 + \dots + \frac{1}{\sqrt{(\alpha-1)(\alpha-2)/2}}A_z^{\alpha-2} + \frac{\alpha}{\sqrt{\alpha(\alpha-1)/2}}A_z^{\alpha-1}) \quad (34)$$

and

$$D_+^\alpha \psi_\alpha = 0 ; \alpha = 1, \dots, N-1. \quad (35)$$

The Gauss's law constraint Eq. (21) is given by

$$\partial_z A_{\bar{z}}^a - \partial_{\bar{z}} A_z^a = Q^a(\psi, \bar{\psi}) - v^a. \quad (36)$$

In the  $CP(1)$  case, we have only one complex  $\psi$  which we parameterize by

$$\bar{\psi} = \rho \exp(i\phi) \quad (37)$$

where  $\rho$  is real and the phase  $\phi$  is a real multi-valued function. Then, Eq. (33) can be solved for the gauge field  $A$  and Eq. (21) reduces to a vortex-type equation;

$$\begin{aligned} A_i &= \epsilon_{ij} \partial_j \log \rho - \partial_i \phi \\ \nabla^2 \log \rho + \epsilon_{ij} \partial_i \partial_j \phi &= \frac{1}{\kappa} \left( v - \frac{1 - \rho^2}{1 + \rho^2} \right). \end{aligned} \quad (38)$$

The derivative term  $\epsilon_{ij} \partial_i \partial_j \phi$  is identically zero except at the zeros of  $\bar{\psi}$  where the multi-valuedness of  $\phi$  results in the Dirac delta function (see, for example, [14]). In the  $CP(2)$  case, we define  $\bar{\psi}_1 = \rho_1 \exp(i\phi_1)$ ,  $\bar{\psi}_2 = \rho_2 \exp(i\phi_2)$  and solve Eq. (33) for  $A^1$  and  $A^2$  so that the resulting vortex-type equation becomes

$$\begin{aligned} A_i^1 &= \epsilon_{ij} \partial_j \log \rho_1 - \partial_i \phi_1 \\ A_i^2 &= \frac{2}{\sqrt{3}} \epsilon_{ij} \partial_j \log \rho_2 - \frac{1}{2} \epsilon_{ij} \partial_j \log \rho_1 - \frac{2}{\sqrt{3}} \partial_i \phi_2 + \frac{1}{2} \partial_i \phi_1 \\ \nabla^2 \log \rho_1 + \epsilon_{ij} \partial_i \partial_j \phi_1 &= \frac{1}{\kappa} \left( v^1 - \frac{1 - \rho_1^2}{1 + \rho_1^2 + \rho_2^2} \right) \\ \frac{2}{\sqrt{3}} \nabla^2 \log \rho_2 + \frac{2}{\sqrt{3}} \epsilon_{ij} \partial_i \partial_j \phi_2 &= \frac{1}{\kappa} \left( \frac{v^1}{2} + v^2 - \frac{1}{2} \frac{1 - \rho_1^2}{1 + \rho_1^2 + \rho_2^2} - \frac{1}{\sqrt{3}} \frac{1 + \rho_1^2 - 2\rho_2^2}{1 + \rho_1^2 + \rho_2^2} \right). \end{aligned} \quad (39)$$

In general for  $CP(N-1)$ , the number of  $\psi$ 's is half the degrees of freedom of  $CP(N-1)$ , that is,  $(N^2 - 1 - (N-1)^2)/2 = N-1$ , so that Eq. (33) can be solved for  $A^a$ ;  $a = 1, \dots, N-1$  and the Gauss's law reduces to the  $N-1$  vortex-type partial differential equations. To our knowledge, these are new vortex-type equations. A numerical analysis suggests that these vortex-type equations indeed possess vortex solutions and show a rich structure depending on the value of  $v^a$ 's. Details on the rotationally symmetric solutions and their properties will appear elsewhere [15]. Here, we only contend that such vortices saturate the bound

$$E = 4\pi|T|. \quad (40)$$



The energy  $E$  and the topological charge  $T$  can be computed from the Hamiltonian

$$E = 4 \int d^2x g_{\alpha\beta} (D_+^\alpha \psi_\alpha \bar{D}_+^\beta \bar{\psi}_\beta + \bar{D}_-^\alpha \psi_\alpha D_-^\beta \bar{\psi}_\beta) \quad (41)$$

where  $g_{\alpha\beta}$  is the Fubini-Study metric on  $CP(N-1)$ ,

$$g_{\alpha\beta} = \frac{(1 + |\psi|^2)\delta_{\alpha\beta} - \bar{\psi}_\alpha \psi_\beta}{(1 + |\psi|^2)^2}, \quad (42)$$

and the expression for the topological charge in this case is

$$T = \frac{1}{\pi} \int d^2x \left\{ g_{\alpha\beta} (\partial\psi_\alpha \bar{\partial}\bar{\psi}_\beta - \bar{\partial}\psi_\alpha \partial\bar{\psi}_\beta) + \frac{1}{4} \epsilon_{ij} \partial_i \left( (v^a - Q^a(\psi, \bar{\psi})) A_j^a \right) \right\}. \quad (43)$$

In general, with an appropriate normalization, this topological charge  $T$  is taken to be integer valued [11]. In the absence of gauge fields, solutions of Eq. (33) and Eq. (35) are simply given by holomorphic or antiholomorphic functions,  $\psi_\alpha \equiv \psi_\alpha(z)$  or  $\psi_\beta \equiv \psi_\beta(\bar{z})$ , and the solutions of the self-dual equation (30) is obtained by substituting these functions into the Eq. (28). This generalizes the Belavin-Polyakov solution for the  $CP(1)$  case [11].

In conclusion, we have obtained a new vortex-type equations by considering the (2+1)-dimensional gauged generalized Heisenberg ferromagnet model coupled with the Chern-Simons gauge fields. It was shown that the Hermitian symmetric space plays an essential role in deriving the Bogomol'nyi bound in addition to the specific choice potential. Our approach allows a systematic generalization of the vortex equation according to each symmetric spaces. It would be an interesting problem to extend our analysis on the vortices to other Hermitian symmetric space  $G/H$ , and study the properties of vortex-type nonlinear equations, e.g. existence of multi-vortices, in the general context of group theory.

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